6: Gates

Saturday, March 20, 2021 4:01 PM

Quantum Operations (gates)

- Reversible operations (unlike measurements)
- Represented by: unitary matrices.

U is unitary iff $UU^\dagger = U^\dagger U = I$

($UU^\dagger$: regular matrix multiplication; $U^\dagger$ = conjugate transpose of U; I: identity $\begin{pmatrix}1&0\\0&1\end{pmatrix}$)

ex: $U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ -i & -1\end{pmatrix}$

$$U^\dagger = (U^T)^* = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ i & -1\end{pmatrix}^* = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ -i & -1\end{pmatrix}$$

$$UU^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ -i & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ -i & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$$

If U acts on $|\psi\rangle$, state becomes $U|\psi\rangle = |\psi'\rangle$ (matrix multiplication)

Bra: $\langle\psi'| = \langle\psi|U^\dagger$

$(AB)^\dagger = B^\dagger A^\dagger$ ($B \leftarrow |\psi\rangle$, $B^\dagger \leftarrow \langle\psi|$)

$U \Rightarrow 2\times 2$ matrix $\Rightarrow$ 1-qubit operation

$U \Rightarrow 4\times 4$ matrix $\Rightarrow$ 2-qubit operation

3 Ways to Represent Quantum Gates

1. Matrix:

ex: $I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $X = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $Y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $Z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

($X \Leftrightarrow 90° \Leftrightarrow$ \)

Paulis

$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$ ($\Leftrightarrow 45°$)

CNOT = $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$ ← 2 qubit gate

Control-Not

2. Action on standard basis

H: $|0\rangle \to |+\rangle$
$|1\rangle \to |-\rangle$

ex ⇒ CNOT $\begin{pmatrix} & \end{pmatrix}\begin{pmatrix} \end{pmatrix} = \begin{pmatrix} \end{pmatrix}$

3. Ketbra Notation

$|\psi\rangle = \begin{pmatrix}a_0\\a_1\end{pmatrix}$ $|\phi\rangle = \begin{pmatrix}b_0\\b_1\end{pmatrix}$

$$|\psi\rangle\langle\phi| = |\psi\rangle\langle\phi| = \begin{pmatrix}a_0\\a_1\end{pmatrix}\begin{pmatrix}b_0^* & b_1^*\end{pmatrix} = \begin{pmatrix}a_0b_0^* & a_0b_1^*\\ a_1b_0^* & a_1b_1^*\end{pmatrix}$$

Outer product

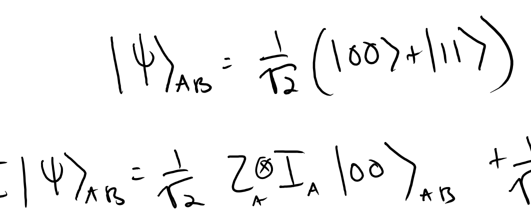
ex: $|+\rangle\langle 0| + |-\rangle\langle 1|$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix} = H$$

$$(|+\rangle\langle 0| + |-\rangle\langle 1|)\,|1\rangle$$
$$= |+\rangle\underbrace{\langle 0|1\rangle}_{0} + |-\rangle\underbrace{\langle 1|1\rangle}_{1} = |-\rangle$$

Unitary on 1 part of 2-qubit State:



Ahyeong applies Z → $U = Z_A \otimes I_B$ ← Barack does nothing

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$Z\otimes I\,|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} Z_A\otimes I_A |00\rangle_{AB} + \frac{1}{\sqrt{2}} Z_A\otimes I_B |11\rangle_{AB}$$

(match up A's, B's.)

$$= \frac{1}{\sqrt{2}}\left(Z|0\rangle_A I|0\rangle_B + Z|1\rangle_A I|1\rangle_B\right)$$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |1\rangle|1\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

Q: $I\otimes Z$ using each method:

- $|00\rangle \to |00\rangle$
  $|01\rangle \to -|01\rangle$
  $|10\rangle \to |10\rangle$
  $|11\rangle \to -|11\rangle$

  Global phase?
  $-|01\rangle$ vs $|01\rangle$
  $I\otimes Z(|00\rangle + |01\rangle) = |00\rangle - |01\rangle$

- Matrix?

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$$

- Ket-bra

$$|00\rangle\langle 00| - |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$$

Questions

Q: If U is unitary and $|\psi\rangle$ is a quantum state is $U|\psi\rangle$ always a quantum state?
- Vector?
- $\langle\psi|\psi\rangle = 1$?

Q: Why is U reversible? (What type of matrix is $U^\dagger$?)

A: Let $|\psi'\rangle = U|\psi\rangle$. $|\psi'\rangle$ is a state if it is properly normalized:

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|I|\psi\rangle = \langle\psi|(I|\psi\rangle) = \langle\psi|\psi\rangle = 1$$

Also $U|\psi\rangle$ is vector with appropriate dimensions

A: If $|\psi\rangle \to U|\psi\rangle$, we can reverse U by applying $U^\dagger$ to get $U^\dagger U|\psi\rangle = |\psi\rangle$. But we can only apply $U^\dagger$ if $U^\dagger$ is unitary. Now $(U^\dagger)^\dagger = U$, so $U^\dagger(U^\dagger)^\dagger = U^\dagger U = I$, so $U^\dagger$ is unitary.